

EXAMPLES OF RICCI-MEAN CURVATURE FLOWS

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ABSTRACT. Let $\pi : \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k)) \rightarrow \mathbb{P}^{n-1}$ be a projective bundle over \mathbb{P}^{n-1} with $1 \leq k \leq n-1$. In this paper, we show that lens space $L(k; 1)(r)$ with radius r embedded in $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ is a self-similar solution, where $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ is endowed with the $U(n)$ -invariant gradient shrinking Ricci soliton structure. We also prove that there exists a pair of critical radii $r_1 < r_2$ which satisfies the following. The lens space $L(k; 1)(r)$ is a self-shrinker if $r < r_2$ and self-expander if $r_2 < r$, and the Ricci-mean curvature flow emanating from $L(k; 1)(r)$ collapses to the zero section of π if $r < r_1$ and to the ∞ -section of π if $r_1 < r$. This gives explicit examples of Ricci-mean curvature flows.

1. BACKGROUND

Let M and N be manifolds, $\mathcal{G} = \{g_t \mid t \in [0, T)\}$ be a smooth 1-parameter family of Riemannian metrics on N and $\mathcal{F} = \{F_t : M \rightarrow N \mid t \in [0, T')\}$ be a smooth 1-parameter family of immersions with $T' \leq T$.

Definition 1.1. The pair $(\mathcal{G}, \mathcal{F})$ is called a *Ricci-mean curvature flow* if it satisfies

$$(1) \quad \begin{aligned} \frac{\partial g_t}{\partial t} &= -2 \operatorname{Ric}(g_t) \\ \frac{\partial F_t}{\partial t} &= H_{g_t}(F_t), \end{aligned}$$

where $H_{g_t}(F_t)$ denotes the mean curvature vector field of F_t calculated with the ambient metric g_t at each time t .

The first equation of (1) is just the Ricci flow equation on N and it does not depend on existence of \mathcal{F} . The second equation of (1) is the mean curvature flow equation though it is affected by the evolution of ambient metrics g_t . This is a coupled flow of the Ricci flow and the mean curvature flow.

The Ricci-mean curvature flow equation has been already appeared in some contexts. For example, Smoczyk [10], Han-Li [4] and Lotay-Pacini [7] consider the Lagrangian mean curvature flow coupled with the Kähler-Ricci flow, and generalize several results which hold in Calabi-Yau manifolds to this moving ambient setting. Other contexts appear in, for example, Lott [8] and Magni-Mantegazza-Tsatis [9]. There is a monotonicity formula for a mean curvature flow in a Euclidean space introduced by Huisken [5]. They generalized it to a Ricci-mean curvature flow coupled with a Ricci flow constructed by a gradient shrinking Ricci soliton.

A *gradient shrinking Ricci soliton* is a pair of a Riemannian manifold (N, g) and a function f on N , called a potential function, which satisfies

$$\operatorname{Ric}(g) + \operatorname{Hess} f = g.$$

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From a given gradient shrinking Ricci soliton (N, g, f) and an arbitrary fixed time $T \in (0, \infty)$ the solution of Ricci flow which survives on $[0, T)$ is constructed by $g_t = 2(T - t)\Phi_t^*g$, where Φ_t is the 1-parameter family of diffeomorphisms of N generated by $\frac{1}{2(T-t)}\nabla f$ with $\Phi_0 = \text{id}$.

Motivated by their works, the author generalized the rescaling procedure due to Huisken to a Ricci-mean mean curvature flow in [11]. It states that if a Ricci-mean curvature flow coupled with a Ricci flow $g_t = 2(T - t)\Phi_t^*g$ develops singularities of type I at the same time T , its rescaling limit is a self-shrinker. The definition of self-shrinkers is the following.

Definition 1.2. An immersion $F : M \rightarrow N$ to a gradient shrinking Ricci soliton (N, g, f) is called a *self-similar solution* with coefficient $\lambda \in \mathbb{R}$ if it satisfies

$$(2) \quad H_g(F) = \lambda \nabla f^\perp.$$

If $\lambda < 0$ or $\lambda > 0$, it is called a *self-shrinker* or *self-expander*, respectively.

If $\lambda = 0$, it is a minimal immersion. Moreover, it also holds that a self-similar solution with coefficient λ is a minimal immersion in N with respect to a conformally rescaled metric $e^{2\lambda f/m}g$, where $m = \dim M$. Hence, self-similar solutions can be considered as a kind of generalization of minimal submanifolds. When (N, g, f) is the Gaussian soliton, that is, the Euclidean space with the standard metric and potential function $f = |x|^2/2$, the equation (2) is written as $H(F) = \lambda x^\perp$. Thus, Definition 1.2 coincides with the ordinary notion of self-similar solutions in the Euclidean space. The original form of Definition 1.2 for general Ricci solitons has been appeared in [8].

It is well-known that a mean curvature flow in a fixed Riemannian manifold is a backward L^2 -gradient flow of the volume functional, and a Ricci flow can be regarded as a gradient flow of Perelman's \mathcal{W} -entropy functional. As mentioned above, Ricci-mean curvature flows have some common properties with mean curvature flows in Euclidean spaces or more generally Ricci flat spaces. However, it becomes unclear that a Ricci-mean curvature flow can be considered as a backward L^2 -gradient flow of some functional. To the authors knowledge, a problem to find an appropriate functional such that its gradient flow is a Ricci-mean curvature flow is still open.

If M and N are compact, for any Riemannian metric g on N and immersion $F : M \rightarrow N$, the short-time existence and uniqueness of the Ricci-mean curvature flow equation (1) with initial condition (g, F) are assured. Actually, first, construct a unique short-time solution of Ricci flow on N with initial metric g , and next, solve the second equation of (1) for short-time with initial immersion F . Hence, there are infinitely many examples of Ricci-mean curvature flows.

However, explicit examples of Ricci-mean curvature flows are not known, so far. In this paper, we consider a lens space $L(k; 1)(r)$ embedded in $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ endowed with a gradient shrinking Ricci soliton structure, and investigate how it moves along the Ricci-mean curvature flow. The analysis is done by reducing PDE (1) to ODE (20). This example shows how the evolution of the ambient metrics affects the motion of submanifolds. To the authors knowledge, this gives a first non-trivial explicit example of Ricci-mean curvature flow, and the author hopes that this example inspires further research of Ricci-mean curvature flows.

Organization of this paper. Section 1 is a background of a study of Ricci-mean curvature flows, and definitions of Ricci-mean curvature flows and self-similar solutions are given in it. In Section 2, we briefly summarize main results of this paper. In Section 3, we review a construction of a gradient shrinking Kähler Ricci soliton structure on $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$. In Section 3, we see that lens spaces $L(k; 1)(r)$ with radius r are naturally embedded in $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ and these are self-similar solutions. In Section 5, we investigate the motion of the Ricci-mean curvature flow emanating from $L(k; 1)(r)$. In Section 6, we compare it to the ordinary mean curvature flow emanating from $L(k; 1)(r)$.

2. MAIN RESULTS

We give a summary of results of this paper in the following. An ambient space is the k -twisted projective-line bundle $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ over \mathbb{P}^{n-1} , where $n \geq 2$ and $1 \leq k \leq n-1$, and we denote it by N_k^n . It can be shown that N_k^n contains $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$ as an open dense subset, and its complement is the disjoint union of S_0 and S_∞ , where these denote the image of 0-section and ∞ -section respectively. The eliminated $\{0\}$ and the point at infinity of $\mathbb{C}^n \setminus \{0\}$ are replaced by S_0 and S_∞ respectively. See Figure 1.

It is known, by Cao [1] and Koiso [6], that there exists a unique $U(n)$ -invariant gradient shrinking Kähler Ricci soliton structure on N_k^n . We denote its Riemannian metric and potential function by g and f , respectively.

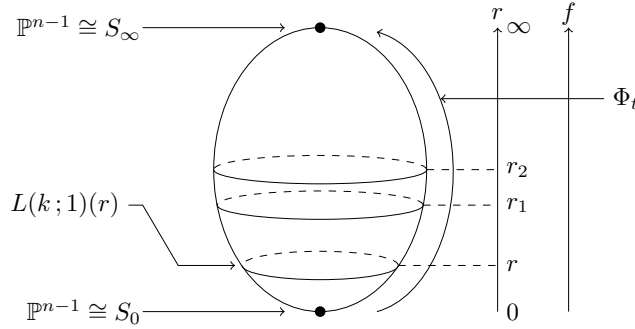


FIGURE 1. N_k^n

For $0 < r < \infty$, we consider the \mathbb{Z}_k quotient of $S^{2n-1}(r)$, the sphere with radius r in \mathbb{C}^n , and denote it by $L(k; 1)(r)$. Actually, it is a lens space and embedded in N_k^n . We denote its inclusion map by

$$\iota_r : L(k; 1)(r) \hookrightarrow N_k^n.$$

Then, the first theorem states that these are self-similar solutions, and whether it is a self-shrinker or self-expander is distinguished by whether its radius is smaller or larger than a critical radius r_2 .

Theorem 2.1. *For each $0 < r < \infty$, the inclusion map $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a compact self-similar solution. Furthermore, there exists a unique radius r_2 which satisfies the following.*

- If $r < r_2$, $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a non-minimal self-shrinker.
- If $r = r_2$, $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a minimal embedding.
- If $r > r_2$, $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a non-minimal self-expander.

Remark 2.2. Especially, Theorem 4.3 claims that there exists a compact self-expander in N_k^n . To contrast with the case that the ambient space is a Euclidean space, we remark that there exists no compact self-expander in \mathbb{R}^n . It can be proved by several ways, for instance, see Proposition 5.3 in [2]. In \mathbb{R}^n , the sphere $S^{n-1}(r)$ is a self-shrinker for every radius r . However, intuitively, we get a self-expander in (N_k^n, ω, f) by taking the radius sufficiently large because of bending and compactifying the neighborhood of $\{\infty\}$ of $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$.

Fix a time $0 < T < \infty$. Then, we will check that the 1-parameter family of diffeomorphisms Φ_t generated by $\frac{1}{2(T-t)}\nabla f$ with $\Phi_0 = \text{id}$ is given by

$$\Phi_t(z) := \left(\frac{T}{T-t} \right)^{\frac{c}{2}} z$$

for $t \in [0, T)$ on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$. Here, c is a positive constant defined in the process to construct the soliton structure, and we skip its explanation here. Then, we obtain a Ricci flow

$$g_t := 2(T-t)\Phi_t^*g$$

which survives on the time interval $[0, T)$. We remark that, since c is positive, Φ_t is expanding and $\Phi_t(z)$ converges to a point in S_∞ as $t \rightarrow T$ if z is not contained in S_0 . See Figure 1.

For a fixed radius r , we take the solution of Ricci-mean curvature flow $F_t : L(k; 1)(r) \rightarrow N_k^n$ along g_t with initial condition $F_0 = \iota_r$. We assume that F_t exists on $[0, T')$ and $T' (= T'(r))$ is the maximal time of existence of the solution. Note that $T' \leq T$ in general. Then, the following is a summary of Theorem 5.5.

Theorem 2.3. *There exists a unique radius r_1 with $r_1 < r_2$ which satisfies the following.*

- $T' = T$ if and only if $r = r_1$.
- If $r \leq r_1$, $F_t : L(k; 1)(r) \rightarrow N_k^n$ collapses to S_0 as $t \rightarrow T'$.
- If $r > r_1$, $F_t : L(k; 1)(r) \rightarrow N_k^n$ collapses to S_∞ as $t \rightarrow T'$.

Theorem 5.5 contains further information about the blow up rate of the solution. Actually, we see that the blow up rate is type I in each case. The above theorem reveals how a lens space $L(k; 1)(r)$ moves and what it converges to by a Ricci-mean curvature flow along the Ricci flow g_t . On the other hand, in section 6, we investigate the evolution of $L(k; 1)(r)$ by the ordinary mean curvature flow in the fixed Riemannian manifold (N_k^n, g) . Then, we prove that if $r < r_2$ ($r > r_2$) it collapses to S_0 (S_∞) in finite time and its blow up rate is also type I. Of course, if $r = r_2$, $L(k; 1)(r)$ does not move since $L(k; 1)(r)$ is minimal. Thus, the critical radius r_1 which determine whether a lens space tends to S_0 -side or S_∞ -side under the Ricci-mean curvature flow is smaller than the minimal radius r_2 . See Figure 1. Here we summarize the situation on Table 1.

3. QUICK REVIEW OF CAO'S CONSTRUCTION

The first example of non-trivial compact gradient shrinking Ricci soliton was found by Koiso [6] and independently by Cao [1], and it is actually a Kähler Ricci

TABLE 1. Ricci-mean curvature flow and mean curvature flow

Ricci-mean curvature flow					
Radius r	$r < r_1$	r_1	$r_1 < r < r_2$	r_2	$r_2 < r$
Maximal time T'	$T' < T$	$T' = T$	$T' < T$		
Collapse to	S_0		S_∞		
blow up rate	Type I				

Mean curvature flow					
Radius r	$r < r_1$	r_1	$r_1 < r < r_2$	r_2	$r_2 < r$
Maximal time T'	$T' < \infty$			$T' = \infty$	$T' < \infty$
Collapse to	S_0			—	S_∞
blow up rate	Type I			—	Type I

soliton. In this section, we quickly review the construction of it following Section 4 in [1] and also Section 7.2 in [3]. Fix integers n, k with $n \geq 2$ and $1 \leq k \leq n-1$, and consider the k -twisted projective-line bundle

$$\pi : \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k)) \rightarrow \mathbb{P}^{n-1}$$

over \mathbb{P}^{n-1} , where $\mathcal{O}(k)$ denotes the k -th tensor power of the hyperplane bundle $\mathcal{O}(1)$ over \mathbb{P}^{n-1} . The gradient shrinking Kähler Ricci soliton is constructed on $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$. Let $(z^1 : \dots : z^n)$ be the homogeneous coordinates on \mathbb{P}^{n-1} , and put $U_j := \{z^j \neq 0\} \subset \mathbb{P}^n$ for $j = 1, \dots, n$. Then $\{U_1, \dots, U_n\}$ gives an open covering of \mathbb{P}^{n-1} , and the transition functions of $\mathcal{O}(k)$ are given by

$$(3) \quad y^i = \left(\frac{z^i}{z^j} \right)^k y^j$$

over $U_i \cap U_j$, where $y^i \in \mathbb{C}$ is the standard coordinate of a fiber of $\mathcal{O}(k)$ over U_i . For a point $(z^1, \dots, z^n) \in \mathbb{C}^n \setminus \{0\}$ with $z^j \neq 0$, we define

$$\psi(z^1, \dots, z^n) := ((z^1 : \dots : z^n), (1 : (z^j)^k)) \in U_j \times \mathbb{P}^1 \cong \pi^{-1}(U_j).$$

This definition is compatible for $(z^1, \dots, z^n) \in \mathbb{C}^n \setminus \{0\}$ with $z^i \neq 0$ and $z^j \neq 0$ since $(1 : (z^j)^k) \in \mathbb{P}(\mathbb{C} \oplus \mathbb{C})$ in the fiber over U_j is identified with $(1 : (z^i)^k) \in \mathbb{P}(\mathbb{C} \oplus \mathbb{C})$ in the fiber in U_i by the relation (3). Hence we have a smooth map

$$\psi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k)).$$

It is clear that $\psi(z^1, \dots, z^n) = \psi(z'^1, \dots, z'^n)$ if and only if $z' = e^{2\pi i \frac{\ell}{k}} z$ for some $\ell \in \mathbb{Z}$. Hence ψ induces an open dense embedding

$$\hat{\psi} : (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k \hookrightarrow \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k)),$$

where the \mathbb{Z}_k -action on $\mathbb{C}^n \setminus \{0\}$ is defined by $([\ell], z) \mapsto e^{2\pi i \frac{\ell}{k}} z$, and the complement of the image of $\hat{\psi}$ is $S_0 \sqcup S_\infty$, where S_0 and S_∞ denote the image of 0-section and ∞ -section of π , respectively.

From now on, we denote $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(k))$ by N_k^n , and we identify $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$ with its image of $\hat{\psi}$. Thus, we have an open dense subset

$$(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k \subset N_k^n.$$

The Kähler Ricci soliton structure is constructed on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, and it actually extends smoothly to S_0 and S_∞ . Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function which satisfies

$$(4) \quad u'(s) > 0 \quad \text{and} \quad u''(s) > 0,$$

and has the following asymptotic expansions

$$(5) \quad \begin{aligned} u(s) &= (n-k)s + a_1 e^{ks} + a_2 e^{2ks} + \cdots \quad (s \rightarrow -\infty) \\ u(s) &= (n+k)s + b_1 e^{-ks} + b_2 e^{-2ks} + \cdots \quad (s \rightarrow \infty) \end{aligned}$$

with $a_1 > 0$ and $b_1 > 0$. Define a $U(n)$ -invariant smooth function $\Phi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ by

$$\Phi(z) := u(s) \quad \text{with} \quad s = \log |z|^2.$$

Since Φ is \mathbb{Z}_k -invariant, it induce a smooth function on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, and we continue to denote it by Φ . By the positivity conditions (4), we get a Kähler form

$$(6) \quad \omega = \sqrt{-1} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta$$

on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, where (z^1, \dots, z^n) is the $(k\text{-to-one})$ global holomorphic coordinates. By the asymptotic conditions (5), Kähler form ω extends smoothly to S_0 and S_∞ , and we get a global Kähler structure on N_k^n . The Ricci form of ω is

$$\text{Ric}(\omega) = -\sqrt{-1} \frac{\partial^2 \log \det(g)}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta,$$

where $g = (g_{\alpha\bar{\beta}})$ is a matrix given by

$$(7) \quad g_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = e^{-s} u'(s) \delta_{\alpha\bar{\beta}} + e^{-2s} \bar{z}^\alpha z^\beta (u''(s) - u'(s))$$

for $s = \log |z|^2$. One can easily check that

$$\begin{aligned} g^{\alpha\bar{\beta}}(z) &= \frac{e^s}{u'(s)} \delta^{\alpha\bar{\beta}} + z^\alpha \bar{z}^\beta \left(\frac{1}{u''(s)} - \frac{1}{u'(s)} \right), \\ \det(g(z)) &= e^{-ns} (u'(s))^{n-1} u''(s). \end{aligned}$$

Define a real valued smooth function $P : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(8) \quad \begin{aligned} P(s) &:= \log (e^{-ns} (u'(s))^{n-1} u''(s)) + u(s) \\ &= -ns + (n-1) \log u'(s) + \log u''(s) + u(s), \end{aligned}$$

and a $U(n)$ -invariant real valued smooth function $f : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ by

$$(9) \quad f(z) := P(s) = \log \det(g(z)) + \Phi(z) \quad \text{with} \quad s = \log |z|^2.$$

Since f is \mathbb{Z}_k -invariant, it induce a smooth function on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, and we continue to denote it by f . Then, we have

$$\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} f = \omega.$$

This equation is just the (1,1)-part of the gradient shrinking Ricci soliton equation

$$(10) \quad \text{Ric} + \text{Hess } f = g,$$

where g is the associated Riemannian metric of ω and Ric is the Ricci 2-tensor of g . Thus, the property that f satisfies (10) is equivalent to that ∇f is a holomorphic vector field. The coefficient of $\partial/\partial z^\alpha$ of ∇f is given by

$$g^{\alpha\bar{\beta}} \frac{\partial f}{\partial \bar{z}^\beta} = g^{\alpha\bar{\beta}} (P'(s)e^{-s}z^\beta) = \frac{P'(s)}{u''(s)}z^\alpha.$$

Hence, ∇f is holomorphic if and only if

$$(11) \quad \frac{P'(s)}{u''(s)} = c$$

for some constant $c \in \mathbb{R}$. Substituting (8) into (11), we have the following third order ODE:

$$(12) \quad \frac{u'''}{u''} + \left(\frac{n-1}{u'} - c \right) u'' = n - u'.$$

Hence, we get a $U(n)$ -invariant gradient shrinking Kähler Ricci soliton structure on N_k^n when we find a solution u of (12) which satisfies condition (4) and (5) for some $c \in \mathbb{R}$. Then, Cao [1] proved the following.

Theorem 3.1 ([1]). *There exists one and only one pair (u, c) so that u and c satisfies (4), (5) and (12). Additionally, it follows that $0 < c < 1$.*

Thus, there exists a unique $U(n)$ -invariant gradient shrinking Kähler Ricci soliton structure on N_k^n , and Kähler form ω and potential function f are written as (6) and (9) on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, respectively.

4. LENS SPACES IN N_k^n AS SELF-SIMILAR SOLUTIONS

In this section we see that a lens space $L(k; 1)(r)$ with radius r is embedded in (N_k^n, ω, f) as a self-similar solution, and whether it is a self-shrinker or self-expander is determined by its radius r . Actually, we prove that there exists the specific radius r_2 such that $L(k; 1)(r)$ is a self-shrinker or self-expander if $r < r_2$ or $r > r_2$, respectively.

Let p, q_1, \dots, q_n be integers such that q_i are coprime to p , and r be a positive constant. Then, the lens space $L(p; q_1, \dots, q_n)(r)$ with radius r is the quotient of $S^{2n-1}(r) \subset \mathbb{C}^n$, the sphere with radius r , by the free \mathbb{Z}_p -action defined by

$$[\ell] \cdot (z^1, \dots, z^n) := (e^{2\pi i \ell \frac{q_1}{p}} z^1, \dots, e^{2\pi i \ell \frac{q_n}{p}} z^n).$$

We restrict ourselves to the case that given integers n and k satisfy $n \geq 2$ and $1 \leq k \leq n-1$. We write

$$L(k; 1)(r) := L(k; \overbrace{1, \dots, 1}^n)(r),$$

for short. It is clear that $L(k; 1)(r)$ is embedded in $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, and $U(n)$ acts on $L(k; 1)(r)$ transitively, since \mathbb{Z}_k -action defined by

$$[\ell] \cdot (z^1, \dots, z^n) := (e^{2\pi i \ell \frac{1}{k}} z^1, \dots, e^{2\pi i \ell \frac{1}{k}} z^n)$$

and $U(n)$ -action commute.

Let (N_k^n, ω) be the unique $U(n)$ -invariant gradient shrinking Kähler Ricci soliton with potential function f given in Theorem 3.1. As explained in Section 3, we have an open dense subset

$$(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k \subset N_k^n.$$

Via this identification, we embed $L(k; 1)(r)$ into N_k^n , and denote its inclusion map by

$$\iota_r : L(k; 1)(r) \hookrightarrow N_k^n.$$

Actually, $L(k; 1)(r)$ is given as a level set of potential function f .

Lemma 4.1. *We have*

$$L(k; 1)(r) = \{f = \gamma\},$$

where $\gamma := P(\log r^2)$ and P is given by (8).

Proof. It is clear that $L(k; 1)(r)$ is contained in $\{f = \gamma\}$ by a relation $f(z) = P(s)$ with $s = \log |z|^2$. To show the converse inclusion, it is sufficient to see that P is strictly increasing. This is true since $P' > 0$ by the equality (11), the positivity condition (4) and a fact that $0 < c < 1$ stated in Theorem 3.1. \square

Since $L(k; 1)(r)$ is a level set of f , the second fundamental form A and the mean curvature vector field H of $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ are given by

$$(13) \quad A(\iota_r) = -\frac{\nabla f}{|\nabla f|^2} \text{Hess } f \quad \text{and} \quad H(\iota_r) = -\frac{\nabla f}{|\nabla f|^2} \text{tr}^\top \text{Hess } f,$$

where ∇f and $\text{Hess } f$ is the gradient and the Hessian of f with respect to the ambient Riemannian metric g , and tr^\top is the trace restricted on $T_p L(k; 1)(r)$ at each point p in $L(k; 1)(r)$. Since $L(k; 1)(r)$ and the Kähler structure on N_k^n are invariant under $U(n)$ -action and it acts transitively on $L(k; 1)(r)$, a function

$$-\frac{1}{|\nabla f|^2} \text{tr}^\top (\text{Hess } f)$$

on $L(k; 1)(r)$ is actually a constant, and we denote the constant by $\lambda(r)$. Thus, the embedding $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a self-similar solution with

$$H(\iota_r) = \lambda(r) \nabla f^\perp.$$

Here we used that ∇f is normal to $L(k; 1)(r)$, that is, $\nabla f^\perp = \nabla f$ actually. The reminder is to determine the sign of $\lambda(r)$. By the $U(n)$ -invariance, it suffices to compute

$$-\frac{1}{|\nabla f|^2} \text{tr}^\top (\text{Hess } f)$$

at a point $p = (r, 0, \dots, 0)$ in $L(k; 1)(r)$. Put $s := \log r^2$ and

$$v_1 := \frac{e^{\frac{s}{2}}}{\sqrt{2u''(s)}} \frac{\partial}{\partial y^1} = \sqrt{-1} \frac{e^{\frac{s}{2}}}{\sqrt{2u''(s)}} \left(\frac{\partial}{\partial z^1} - \frac{\partial}{\partial \bar{z}^1} \right).$$

Furthermore, put

$$\begin{aligned} w_\alpha &:= \frac{e^{\frac{s}{2}}}{\sqrt{2u'(s)}} \frac{\partial}{\partial x^\alpha} = \frac{e^{\frac{s}{2}}}{\sqrt{2u'(s)}} \left(\frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \bar{z}^\alpha} \right) \\ Jw_\alpha &:= \frac{e^{\frac{s}{2}}}{\sqrt{2u'(s)}} \frac{\partial}{\partial y^\alpha} = \sqrt{-1} \frac{e^{\frac{s}{2}}}{\sqrt{2u'(s)}} \left(\frac{\partial}{\partial z^\alpha} - \frac{\partial}{\partial \bar{z}^\alpha} \right), \end{aligned}$$

for $\alpha = 2, \dots, n$. Then, by (7), one can check that $\{v_1, w_2, Jw_2, \dots, w_n, Jw_n\}$ is an orthonormal basis of $T_p L(k; 1)(r)$ at $p = (r, 0, \dots, 0)$. Here, we have

$$(14) \quad \begin{aligned} \text{Hess } f(v_1, v_1) &= \frac{e^s}{u''(s)} \frac{\partial^2 f}{\partial z^1 \partial \bar{z}^1}(p) = \frac{P''(s)}{u''(s)} \\ \text{Hess } f(w_\alpha, w_\alpha) &= \text{Hess } f(Jw_\alpha, Jw_\alpha) = \frac{e^s}{u'(s)} \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\alpha}(p) = \frac{P'(s)}{u'(s)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \text{tr}^\top \text{Hess } f &= \text{Hess } f(v_1, v_1) + \sum_{\alpha=2}^n \text{Hess } f(w_\alpha, w_\alpha) + \sum_{\alpha=2}^n \text{Hess } f(Jw_\alpha, Jw_\alpha) \\ &= \frac{P''(s)}{u''(s)} + 2(n-1) \frac{P'(s)}{u'(s)}. \end{aligned}$$

By $P' = cu''$, we have

$$\begin{aligned} \frac{P''}{u''} + 2(n-1) \frac{P'}{u'} &= c \frac{u'''}{u''} + 2c(n-1) \frac{u''}{u'} \\ &= c \left(\left(\frac{u'''}{u''} + (n-1) \frac{u''}{u'} \right) + (n-1) \frac{u''}{u'} \right) \\ &= c \left(n - u' + cu'' + (n-1) \frac{u''}{u'} \right), \end{aligned}$$

where we used ODE (12) in the last equality. Furthermore, It is clear that

$$(15) \quad \nabla f = ce^{\frac{s}{2}} \frac{\partial}{\partial x^1} \quad \text{and} \quad |\nabla f|^2 = 2c^2 u''(s).$$

at $p = (r, 0, \dots, 0) \in L(k; 1)(r)$. Thus, we have

$$\lambda(r) = \frac{-1}{2cu''(s)} \left(n - u'(s) + cu''(s) + (n-1) \frac{u''(s)}{u'(s)} \right),$$

where $s = \log r^2$.

To capture the behavior of $\lambda(r)$, we need the following lemma. The radius r_2 in the statement (2) of the following lemma is needed to determine whether $L(k; 1)(r)$ is a self-shrinker or a self-expander, and r_1 determines whether the Ricci-mean curvature flow of $L(k; 1)(r)$ converges to S_0 or S_∞ .

Lemma 4.2.

(1) *It holds that*

$$\begin{aligned} \lambda(r) &\rightarrow -\infty \quad \text{and} \quad \lambda(r) = \mathcal{O}(r^{-2k}) \quad \text{as} \quad r \rightarrow 0, \\ \lambda(r) &\rightarrow \infty \quad \text{and} \quad \lambda(r) = \mathcal{O}(r^{2k}) \quad \text{as} \quad r \rightarrow \infty. \end{aligned}$$

(2) *There exists a unique pair of radii $r_1 < r_2$ which satisfies the following.*

- $\lambda(r) \in (-\infty, -1)$ for $r \in (0, r_1)$.
- $\lambda(r_1) = -1$.
- $\lambda(r) \in (-1, 0)$ for $r \in (r_1, r_2)$.
- $\lambda(r_2) = 0$.
- $\lambda(r) \in (0, \infty)$ for $r \in (r_2, \infty)$.

Proof. By the asymptotic conditions (5), we have

$$\begin{aligned} n - u'(s) + cu''(s) + (n-1)\frac{u''(s)}{u'(s)} &\rightarrow k \quad (s \rightarrow -\infty) \\ n - u'(s) + cu''(s) + (n-1)\frac{u''(s)}{u'(s)} &\rightarrow -k \quad (s \rightarrow \infty), \end{aligned}$$

and also have

$$\begin{aligned} u''(s) &\rightarrow 0 \quad \text{and} \quad u''(s) = \mathcal{O}(e^{ks}) \quad (s \rightarrow -\infty) \\ u''(s) &\rightarrow 0 \quad \text{and} \quad u''(s) = \mathcal{O}(e^{-ks}) \quad (s \rightarrow \infty). \end{aligned}$$

Thus, we have proved the statement (1).

To prove the statement (2), we will prove that the derivative of $\lambda(r)$ at r such that $\lambda(r) = -1$ or $\lambda(r) = 0$ is positive. Then, combining the statement (1), this implies immediately that $\lambda(r)$ takes the value -1 and 0 only once.

Define $\Lambda(s)$ by

$$\Lambda(s) := \lambda(r) = \frac{-1}{2cu''(s)} \left(n - u'(s) + cu''(s) + (n-1)\frac{u''(s)}{u'(s)} \right)$$

with $s = \log r^2$. Then, we have

$$\frac{d}{dr}\lambda(r) = 2e^{-\frac{s}{2}} \frac{d}{ds}\Lambda(s).$$

Hence, the positivity of $d\lambda/dr$ is equivalent to the positivity of $d\Lambda/ds$. By a straightforward computation, we have

$$\begin{aligned} \frac{d}{ds}\Lambda(s) &= \frac{-1}{2cu''(s)} \left(-u''(s) + cu'''(s) \right. \\ &\quad \left. + (n-1)\frac{u'''(s)}{u'(s)} - (n-1)\frac{(u''(s))^2}{(u'(s))^2} \right) - \Lambda(s)\frac{u'''(s)}{u''(s)} \\ (16) \quad &= \frac{1}{2c} + \frac{(n-1)u''(s)}{2c(u'(s))^2} \\ &\quad + \frac{1}{2cu'(s)} \left(-c(1 + 2\Lambda(s))u'(s) - (n-1) \right) \frac{u'''(s)}{u''(s)}. \end{aligned}$$

By ODE (12), we have

$$\begin{aligned} \frac{u'''(s)}{u''(s)} &= n - u'(s) - \left(\frac{n-1}{u'(s)} - c \right) u''(s) \\ (17) \quad &= \left(n - u'(s) + cu''(s) + (n-1)\frac{u''(s)}{u'(s)} \right) - 2(n-1)\frac{u''(s)}{u'(s)} \\ &= -2cu''(s)\Lambda(s) - 2(n-1)\frac{u''(s)}{u'(s)} \\ &= 2 \left(-c\Lambda(s)u'(s) - (n-1) \right) \frac{u''(s)}{u'(s)}. \end{aligned}$$

Substituting (17) into (16), we have

$$\begin{aligned} \frac{d}{ds}\Lambda(s) &= \frac{1}{2c} + \frac{(n-1)u''(s)}{2c(u'(s))^2} \\ &\quad + \frac{1}{c} \left(-c(1+2\Lambda(s))u'(s) - (n-1) \right) \left(-c\Lambda(s)u'(s) - (n-1) \right) \frac{u''(s)}{(u'(s))^2}. \end{aligned}$$

We remark that since $c, u'' > 0$,

$$\frac{1}{2c} + \frac{(n-1)u''(s)}{2c(u'(s))^2} > 0.$$

Since $\Lambda(s) \rightarrow -\infty$ as $s \rightarrow -\infty$ and $\Lambda(s) \rightarrow \infty$ as $s \rightarrow \infty$, there exists an $s \in \mathbb{R}$ such that $\Lambda(s) = 0$, and for such s we have

$$\begin{aligned} &\frac{1}{c} \left(-c(1+2\Lambda(s))u'(s) - (n-1) \right) \left(-c\Lambda(s)u'(s) - (n-1) \right) \frac{u''(s)}{(u'(s))^2} \\ &= \frac{n-1}{c} \left(cu'(s) + (n-1) \right) \frac{u''(s)}{(u'(s))^2} > 0, \end{aligned}$$

since $c, u', u'' > 0$. Thus, we have proved that

$$\frac{d}{ds}\Lambda(s) > \frac{1}{2c} + \frac{(n-1)u''(s)}{2c(u'(s))^2} > 0$$

for s such that $\Lambda(s) = 0$. Similarly, there exists another $s \in \mathbb{R}$ such that $\Lambda(s) = -1$, and for such s we have

$$\begin{aligned} &\frac{1}{c} \left(-c(1+2\Lambda(s))u'(s) - (n-1) \right) \left(-c\Lambda(s)u'(s) - (n-1) \right) \frac{u''(s)}{(u'(s))^2} \\ &= \frac{1}{c} \left(cu'(s) - (n-1) \right)^2 \frac{u''(s)}{(u'(s))^2} \geq 0 \end{aligned}$$

since $c, u'' > 0$. Thus, we have proved that

$$\frac{d}{ds}\Lambda(s) \geq \frac{1}{2c} + \frac{(n-1)u''(s)}{2c(u'(s))^2} > 0$$

for s such that $\Lambda(s) = -1$. Consequently, we have proved that

$$\frac{d}{dr}\lambda(r) > 0$$

for r such that $\lambda(r) = 0$ or $\lambda(r) = -1$. By this property and the statement (1), the statement (2) follows. \square

By Lemma 4.2, we have proved the following. This is the same as Theorem 2.1.

Theorem 4.3. *For every $0 < r < \infty$, the embedding*

$$\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$$

is a compact self-similar solution with

$$H(\iota_r) = \lambda(r)\nabla f^\perp,$$

and there exists the unique radius r_2 such that $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ is a non-minimal self-shrinker, minimal submanifold or non-minimal self-expander when $r < r_2$, $r = r_2$ or $r_2 < r$, respectively.

For the following sections, here we compute the norm of $A(\iota_r)$. It is easy to see that $A(\iota_r)$ is diagonalized by the orthonormal basis $\{v_1, w_2, Jw_2, \dots, w_n, Jw_n\}$. Hence, we have

$$|A(\iota_r)|^2 = |A(v_1, v_1)|^2 + \sum_{\alpha=2}^n |A(w_\alpha, w_\alpha)|^2 + \sum_{\alpha=2}^n |A(Jw_\alpha, Jw_\alpha)|^2.$$

By (13), (14) and (15), with $s = \log r^2$, we have

$$|A(\iota_r)|^2 = \frac{1}{2c^2 u''(s)} \left(\left(\frac{P''(s)}{u''(s)} \right)^2 + 2(n-1) \left(\frac{P'(s)}{u'(s)} \right)^2 \right).$$

By $P' = cu''$ and ODE (12), we have

$$|A(\iota_r)|^2 = \frac{1}{2u''(s)} \left(\left(n - u'(s) - \left(\frac{n-1}{u'(s)} - c \right) u''(s) \right)^2 + 2(n-1) \left(\frac{u''(s)}{u'(s)} \right)^2 \right).$$

By a similar argument of the proof of the statement (1) of Lemma 4.2, we can prove the following.

Lemma 4.4. *It holds that*

$$\begin{aligned} |A(\iota_r)|^2 &\rightarrow \infty \quad \text{and} \quad |A(\iota_r)|^2 = \mathcal{O}(r^{-2k}) \quad \text{as} \quad r \rightarrow 0, \\ |A(\iota_r)|^2 &\rightarrow \infty \quad \text{and} \quad |A(\iota_r)|^2 = \mathcal{O}(r^{2k}) \quad \text{as} \quad r \rightarrow \infty. \end{aligned}$$

5. THE MOTION BY RICCI-MEAN CURVATURE FLOW OF A LENS SPACE IN N_k^n

In this section we observe how a lens space $L(k; 1)(r)$ in N_k^n moves by Ricci-mean curvature flow and what it converges to. Continuing Section 4, let (N_k^n, ω) be a unique $U(n)$ -invariant gradient shrinking Kähler Ricci soliton with potential function f given in Theorem 3.1. Then we have

$$\nabla f = cr \frac{\partial}{\partial r},$$

where $r = |z|$. Fix $T \in (0, \infty)$. One can easily see that

$$\Phi_t : (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k \rightarrow (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$$

defined by

$$\Phi_t(z) := \kappa(t)z \quad \text{with} \quad \kappa(t) := \left(\frac{T}{T-t} \right)^{\frac{c}{2}}$$

for $t \in [0, T)$ is the 1-parameter family of automorphisms of N_k^n generated by $\frac{1}{2(T-t)} \nabla f$ with $\Phi_0 = \text{id}$. Then, it follows that $g_t := 2(T-t)\Phi_t^* g$ satisfies the Ricci flow equation:

$$\frac{\partial}{\partial t} g_t = -2\text{Ric}(g_t).$$

Fix $r \in (0, \infty)$ and let $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ be a lens space with radius r and

$$F_t : L(k; 1)(r) \rightarrow N_k^n \quad (t \in [0, T'))$$

be the solution of Ricci-mean curvature flow along $g_t = 2(T-t)\Phi_t^* g$ with initial condition $F_0 = \iota_r$. We assume that $T' (= T'(r))$ is the maximal time of existence of the solution. By rotational symmetry of $L(k; 1)(r) \subset N_k^n$, the solution F_t is written as

$$F_t(p) := h(t)p,$$

by some positive smooth function $h : [0, T') \rightarrow \mathbb{R}$. Then the Ricci-mean curvature flow equation for F_t is reduced to an ODE for $h(t)$.

Proposition 5.1. *The 1-parameter family of immersions F_t is the solution of the Ricci-mean curvature flow coupled with g_t with initial condition $F_0 = \iota_r$ if and only if the positive smooth function $h : [0, T') \rightarrow \mathbb{R}$ satisfies the following ODE with initial condition:*

$$(18) \quad \begin{aligned} h(0) &= 1 \\ \frac{h'(t)}{h(t)} &= \frac{c}{2(T-t)} \lambda(\kappa(t)h(t)r), \end{aligned}$$

where λ and κ are given functions.

Proof. Recall that the Ricci-mean curvature flow equation is

$$\frac{\partial}{\partial t} F_t = H_{g_t}(F_t),$$

where $H_{g_t}(F_t)$ is the mean curvature vector field of F_t computed with the Riemannian metric $g_t = 2(T-t)\Phi_t^*g$. It is easy to see that

$$H_{g_t}(F_t) = \frac{1}{2(T-t)} H(\iota_{\kappa(t)h(t)r}) = \frac{c}{2(T-t)} \lambda(\kappa(t)h(t)r) \left(r \frac{\partial}{\partial r} \right),$$

where H without subscript g_t denotes the mean curvature vector field with respect to the original ambient metric g . Since

$$\frac{\partial}{\partial t} F_t = \frac{h'(t)}{h(t)} \left(r \frac{\partial}{\partial r} \right),$$

the proposition is proved. □

Put

$$(19) \quad R(t) := \kappa(t)h(t)r = \left(\frac{T}{T-t} \right)^{\frac{c}{2}} h(t)r.$$

Then, we have

$$\frac{R'(t)}{R(t)} = \frac{c}{2(T-t)} + \frac{h'(t)}{h(t)}.$$

Thus, we have the following.

Lemma 5.2. *The ODE (18) for $h(t)$ with initial condition is equivalent to the following ODE for $R(t)$ with initial condition:*

$$(20) \quad \begin{aligned} R(0) &= r \\ \frac{R'(t)}{R(t)} &= \frac{c}{2(T-t)} \left(\lambda(R(t)) + 1 \right). \end{aligned}$$

Therefore, analysis of the motion of Ricci-mean curvature flow emanating from $L(k; 1)(r)$ is reduced to the analysis of $R(t)$. Let r_1 be the specific radius introduced in Lemma 4.2. Then, we have the following.

Lemma 5.3.

- (1) *If $r < r_1$, then $T' < T$ and the solution $R(t)$ of (20) satisfies $R(t) \rightarrow 0$ and $h(t) \rightarrow 0$ as $t \rightarrow T'$. Furthermore, we have*

$$(R(t))^{-2k} = \mathcal{O} \left(\frac{1}{T' - t} \right) \quad \text{as } t \rightarrow T'.$$

- (2) If $r = r_1$, then $T' = T$ and $R(t) = r_1$ is the stationary solution of (20) and $h(t) = \kappa^{-1}(t) \rightarrow 0$ as $t \rightarrow T$.
- (3) If $r_1 < r$, then $T' < T$ and the solution $R(t)$ of (20) satisfies $R(t) \rightarrow \infty$ and $h(t) \rightarrow \infty$ as $t \rightarrow T'$. Furthermore, we have

$$(R(t))^{2k} = \mathcal{O}\left(\frac{1}{T' - t}\right) \quad \text{as } t \rightarrow T'.$$

Proof. The proof is done by an ordinary argument for the bifurcation phenomenon of an ODE. First, we prove the statement (1). Assume that $r < r_1$. In this case, by Lemma 4.2, there exists a constant $\alpha = \alpha(r) < -1$ such that $\lambda(r) \leq \alpha$ for all $r \in (0, r_0]$. At $t = 0$, we have $R'(0) < 0$ by OED (20). If there exists some $t_0 \in (0, T')$ such that $R(t_0) = r$, it follows that

$$R'(t_0) = \frac{c}{2(T - t_0)} (\lambda(r) + 1) r < 0.$$

This means that $R(t) \in (0, r]$ for all $t \in [0, T')$ and $R(t)$ is monotonically decreasing. By ODE (20), we have

$$(21) \quad \frac{R'(t)}{R(t)(\lambda(R(t)) + 1)} = \frac{c}{2(T - t)},$$

and integrating both sides from $t = 0$ to $t = T' - 0$ we have

$$(22) \quad \int_r^{R(T'-0)} \frac{1}{R(\lambda(R) + 1)} dR = \int_0^{T'-0} \frac{c}{2(T - t)} dt.$$

By (1) of Lemma 4.2, we have

$$(23) \quad \frac{1}{R(\lambda(R) + 1)} = \mathcal{O}(R^{2k-1}) \quad \text{as } R \rightarrow 0.$$

Thus, the left hand side of (22) is integrable, and we have proved that $T' < T$. If $\lim_{t \rightarrow T'} R(t) > 0$ then $R'(t)$ is bounded as $t \rightarrow T'$ by ODE (20), and this contradicts that T' is the maximal time of existence of the solution. Thus, it holds that $R(t) \rightarrow 0$ as $t \rightarrow T'$. Integrating both sides of (21) from t to T' and combining the estimate of order (23) and $R(t) \rightarrow 0$ as $t \rightarrow T'$, we have

$$C(R(t))^{2k} \geq \int_t^{T'} \frac{c}{2(T - t)} dt \geq \frac{c}{2T} (T' - t)$$

with some constant $C > 0$ and t sufficiently close to T' . Thus, we have

$$(R(t))^{-2k} = \mathcal{O}\left(\frac{1}{T' - t}\right) \quad \text{as } t \rightarrow T'.$$

Since $R(t) = \kappa(t)h(t)r$, $R(t) \rightarrow 0$ as $t \rightarrow T'$ and $\kappa(t)$ is bounded on $[0, T')$, it holds that $h(t) \rightarrow 0$ as $t \rightarrow T'$. Hence, we have proved the statement (1).

The statement (2) is clear since $\lambda(r_1) + 1 = 0$.

Finally, we prove the statement (3). The argument is very similar to the proof of the statement (1). Assume that $r_1 < r$. In this case, by Lemma 4.2, there exists a constant $\alpha = \alpha(r_0) > -1$ such that $\lambda(r) \geq \alpha$ for all $r \in [r_0, \infty)$. At $t = 0$, we have $R'(0) > 0$ by OED (20). If there exists some $t_0 \in (0, T')$ such that $R(t_0) = r$, it follows that

$$R'(t_0) = \frac{c}{2(T - t_0)} (\lambda(r) + 1) r > 0.$$

This means that $R(t) \in [r, \infty)$ for all $t \in [0, T')$ and $R(t)$ is monotonically increasing. By (1) of Lemma 4.2, we have

$$(24) \quad \frac{1}{R(\lambda(R) + 1)} = \mathcal{O}(R^{-2k-1}) \quad \text{as } R \rightarrow \infty.$$

Thus, the left hand side of (22) is integrable, and we have proved that $T' < T$. If $\lim_{t \rightarrow T'} R(t) < \infty$ then $R'(t)$ is bounded as $t \rightarrow T'$ by ODE (20), and this contradicts that T' is the maximal time of existence of the solution. Thus, it holds that $R(t) \rightarrow \infty$ as $t \rightarrow T'$. Integrating both sides of (21) from t to T' and combining the estimate of order (24) and $R(t) \rightarrow \infty$ as $t \rightarrow T'$, we have

$$C(R(t))^{-2k} \geq \int_t^{T'} \frac{c}{2(T-t)} dt = \frac{c}{2T}(T' - t)$$

with some constant $C > 0$ and t sufficient close to T' . Thus, we have

$$(R(t))^{2k} = \mathcal{O}\left(\frac{1}{T' - t}\right) \quad \text{as } t \rightarrow T'.$$

Since $R(t) = \kappa(t)h(t)r$, $R(t) \rightarrow \infty$ as $t \rightarrow T'$ and $\kappa(t)$ is bounded on $[0, T')$, it holds that $h(t) \rightarrow \infty$ as $t \rightarrow T'$. Hence, we have proved the statement (3). \square

Remark 5.4. In the case $r < r_1$, by integrating both sides of (21) from $t = 0$ to $t = T'$ and straightforward computation, the maximal time $T' (= T'(r))$ is explicitly given as

$$T' = T - T \exp\left(\frac{2}{c} \int_0^r \frac{1}{R(\lambda(R) + 1)} dR\right).$$

Similarly, in the case $r_1 < r$, the maximal time $T' (= T'(r))$ is explicitly given as

$$T' = T - T \exp\left(\frac{-2}{c} \int_r^\infty \frac{1}{R(\lambda(R) + 1)} dR\right).$$

By Lemma 5.3, we can prove the following theorem. This contains Theorem 2.3.

Theorem 5.5.

- (1) If $r < r_1$, then $T' < T$ and $F_t : L(k; 1)(r) \rightarrow N_k^n$ converges pointwise to S_0 as $t \rightarrow T'$. Furthermore, we have $|A_{g_t}(F_t)|_{g_t}^2 \rightarrow \infty$ as $t \rightarrow T'$ and there exists some constant $C > 0$ such that

$$|A_{g_t}(F_t)|_{g_t}^2 \leq \frac{C}{T' - t} \quad \text{on } [0, T').$$

- (2) If $r = r_1$, then $T' = T$ and $F_t : L(k; 1)(r) \rightarrow N_k^n$ is given by $F_t(p) = \kappa^{-1}(t)p$ and converges pointwise to S_0 as $t \rightarrow T$. Furthermore, we have $|A_{g_t}(F_t)|_{g_t}^2 \rightarrow \infty$ as $t \rightarrow T$ and there exists some constant $C > 0$ such that

$$|A_{g_t}(F_t)|_{g_t}^2 = \frac{C}{T' - t} \quad \text{on } [0, T').$$

- (3) If $r_1 < r$, then $T' < T$ and $F_t : L(k; 1)(r) \rightarrow N_k^n$ converges pointwise to S_∞ as $t \rightarrow T'$. Furthermore we have $|A_{g_t}(F_t)|_{g_t}^2 \rightarrow \infty$ as $t \rightarrow T'$ and there exists some constant $C > 0$ such that

$$|A_{g_t}(F_t)|_{g_t}^2 \leq \frac{C}{T' - t} \quad \text{on } [0, T').$$

Proof. It is easy to see that

$$(25) \quad |A_{g_t}(F_t)|_{g_t}^2 = \frac{1}{2(T-t)} |A(\iota_{\kappa(t)h(t)r})|^2 = \frac{1}{2(T-t)} |A(\iota_{R(t)})|^2,$$

where $|A_{g_t}|_{g_t}$ and $|A|$ is the norm of the second fundamental form with respect to the ambient metric g_t and g , respectively. In the case (1), we have $h(t) \rightarrow 0$ as $t \rightarrow T'$ by Lemma 5.3. Let $p = (z^1, \dots, z^n)$ be a point in $L(k; 1)(r) \subset (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$ and assume that $z^j \neq 0$ for some j . Then, $F_t(p) = h(t)p$ is identified with

$$((h(t)z^1 : \dots : h(t)z^n), (1 : (h(t)z^j)^k)) = ((z^1 : \dots : z^n), (1 : (h(t)z^j)^k))$$

in $U_j \times \mathbb{P}^1$ via ψ . Hence, it is clear that $F_t : L(k; 1)(r) \rightarrow N_k^n$ converges pointwise to S_0 , the 0-section, as $t \rightarrow T'$. By the formula (25) and Lemma 4.4 and Lemma 5.3, the remaining of the statement (1) is clear. The proof of (3) is similar. In the case (2), it is enough to put $C := 2|A(\iota_{r_1})|^2$. \square

Remark 5.6. Consider the standard Hopf fibration $S^{2n-1}(r) \rightarrow \mathbb{P}^{n-1}$. When $r \rightarrow 0$, the total space $S^{2n-1}(r)$ collapses to \mathbb{P}^{n-1} and this collapsing is just caused by degeneration of S^1 -fiber. From this view point, the picture of the collapsing of $L(k; 1)(r)$ to S_0 or S_∞ (these are diffeomorphic to \mathbb{P}^{n-1}) is considered as a \mathbb{Z}_k -quotient analog of the collapsing of the Hopf fibration.

6. THE MOTION BY MEAN CURVATURE FLOW OF A LENS SPACE IN N_k^n

In this section, we observe how a lens space in N_k^n moves by mean curvature flow. As in Section 5, let (N_k^n, ω) be a unique $U(n)$ -invariant gradient shrinking Kähler Ricci soliton with potential function f given in Theorem 3.1, and for a given $r > 0$ let $\iota_r : L(k; 1)(r) \hookrightarrow N_k^n$ be a lens space with radius r . Then, by rotationally symmetry, the solution

$$F_t : L(k; 1)(r) \rightarrow N_k^n$$

of the mean curvature flow equation

$$\frac{\partial}{\partial t} F_t = H(F_t)$$

is given by

$$F_t(p) = h(t)p$$

with some positive smooth function $h : [0, T') \rightarrow \mathbb{R}$ which satisfies the following ODE with initial condition:

$$(26) \quad \begin{aligned} h(0) &= 1 \\ \frac{h'(t)}{h(t)} &= c\lambda(h(t)r). \end{aligned}$$

We remark that this is an autonomous differential equation. We assume that $T' (= T'(r))$ is the maximal time of existence of the solution. Let r_2 be the specific radius introduced in Lemma 4.2. Then, by similar arguments as the proof of Lemma 5.3, one can prove the following.

Lemma 6.1.

(1) If $r < r_2$, then $T' < \infty$ and the solution $h(t)$ of (26) satisfies

$$h(t) \rightarrow 0 \quad \text{and} \quad (h(t))^{-2k} = \mathcal{O}\left(\frac{1}{T' - t}\right) \quad \text{as } t \rightarrow T'.$$

(2) If $r = r_2$, then $T' = \infty$ and $h(t) = 1$ is the stationary solution of (26).

(3) If $r_2 < r$, then $T' < \infty$ and the solution $h(t)$ of (26) satisfies

$$h(t) \rightarrow \infty \quad \text{and} \quad (h(t))^{2k} = \mathcal{O}\left(\frac{1}{T' - t}\right) \quad \text{as } t \rightarrow T'.$$

Furthermore, as the proof of Theorem 5.5, combining Lemma 4.4 and Lemma 6.1, we can prove the following.

Theorem 6.2.

- (1) If $r < r_2$, then $T' < \infty$ and $F_t : L(k; 1)(r) \rightarrow N_k^n$ converges pointwise to S_0 as $t \rightarrow T'$. Furthermore, we have $|A(F_t)|^2 \rightarrow \infty$ as $t \rightarrow T'$ and there exists some constant $C > 0$ such that

$$|A(F_t)|^2 \leq \frac{C}{T' - t} \quad \text{on } [0, T'),$$

that is, F_t develops singularities of type I.

- (2) If $r = r_2$, then $F_t \equiv F_0 : L(k; 1)(r) \rightarrow N_k^n$ ($t \in [0, \infty)$) is the stationary solution of the mean curvature flow since F_0 is a minimal immersion.
- (3) If $r_2 < r$, then $T' < \infty$ and $F_t : L(k; 1)(r) \rightarrow N_k^n$ converges pointwise to S_∞ as $t \rightarrow T'$. Furthermore we have $|A(F_t)|^2 \rightarrow \infty$ as $t \rightarrow T'$ and there exists some constant $C > 0$ such that

$$|A(F_t)|^2 \leq \frac{C}{T' - t} \quad \text{on } [0, T'),$$

that is, F_t develops singularities of type I.

Remark 6.3. As Remark 5.4, the maximal time $T' (= T'(r))$ is explicitly given as follows. When $r < r_2$,

$$T' = \frac{-1}{c} \int_0^r \frac{1}{r\lambda(r)} dr,$$

and when $r_2 < r$,

$$T' = \frac{1}{c} \int_r^\infty \frac{1}{r\lambda(r)} dr.$$

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